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No horizons in pp-waves

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Abstract: We argue that pp-wave backgrounds can not admit event horizons. We also comment on pp-wave generalizations which would admit horizons and show that there exists a black string solution which asymptotes to a five dimensional plane wave background.

Keywords: Penrose limit and pp-wave background, Classical Theories of Gravity, Black Holes.

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Contents

1. Introduction

Plane wave[1] spacetimes have a special importance in theoretical physics. In general relativity, they form simple solutions to Einstein's equations with many curious properties. They can be thought of as arising from the so-called Penrose limit [1] of any spacetime, which essentially consists of zooming in onto any null geodesic in that spacetime. Being a subset of pp-waves, they admit a covariantly constant null Killing field, which in turn implies that all curvature invariants vanish. Nevertheless, they are distinct from flat spacetime and their structure is much richer. Interestingly, as shown by Penrose in [2], these spacetimes are not globally hyperbolic, so that there exists no Cauchy hypersurface from which a causal evolution would cover the entire spacetime. This automatically implies that even the causal structure of plane waves is different from that of flat spacetime.

pp-wave spacetimes are especially important within the context of string theory. This is because they yield exact classical backgrounds for string theory, since all curvature invariants, and therefore all α' corrections, vanish [3, 4]. Hence the pp-wave spacetimes correspond to exact conformal field theories. Because of this fact, they provide much-needed examples of classical solutions in string theory, which can in turn be used as toy models for studying its structure and properties. Plane waves happen to be even simpler, for the action in light cone gauge is quadratic.

[1]To avoid any confusion later, let us clarify the terminology from the very outset: *pp-waves* (or "plane-fronted waves with parallel rays") are all spacetimes with covariantly constant null Killing field; *plane waves* are a subset of these which have in addition an extra "planar" symmetry along the wavefronts. These are in fact the spacetimes that much of the recent literature discussing Penrose limits has been calling "pp-waves".

While this fact has been appreciated for some time [3, 4], only recently have plane waves received significant attention, mainly initiated by the work of Berenstein, Maldacena, and Nastase (BMN) [5], based on the AdS/CFT correspondence [6, 7, 8, 9]. These authors proposed a very interesting solvable model of string theory in Ramond-Ramond backgrounds by taking the Penrose limit of $AdS_5 \times \mathbf{S}^5$ spacetime [5, 10], the holographic dual of $d = 4$, $\mathcal{N} = 4$ Super-Yang-Mills theory. This limit corresponds to a particular sector of the gauge theory, with large dimensions of operators and large charges, but with a finite difference between the charges and the dimensions. Part of the importance of this result stems from the fact that since the dual background is exactly solvable as a string theory, we can claim to have understood, at least in principle, this particular sector of the gauge theory.

The BMN "correspondence" has since been examined and generalized by many authors and Penrose limits of various supergravity solutions have been considered in the recent literature. One interesting avenue for exploration concerns the addition of black holes into the plane wave spacetime. Naively, this might correspond to "thermalizing" the high energy sector of the gauge theory. While this generalization is rather suggestive and follows in close analogy with the corresponding developments in the AdS/CFT correspondence, where adding a large Schwarzschild black hole into AdS corresponds to thermalizing the gauge theory, no concrete solutions or understanding have yet been reached. Partly, this sector of gauge theory does not yet stand on its own as a well-defined theory without invoking the limit; but more importantly, no appropriate black hole solution has yet been found. In the case of the maximally supersymmetric homogeneous plane-wave discussed by BMN, exact quantization of the light-cone string hamiltonian helps in the analysis of the thermal partition function [11, 12, 13].

One might naively hope to obtain a black hole by taking an appropriate Penrose limit of a more general spacetime. In this paper, we argue that this is *not* possible. In particular, we show that no plane wave can admit event horizons. We will in fact make the stronger claim that no pp-waves can admit event horizons. While the latter may not seem as interesting in the context of the recent excitement about Penrose limits, it is nevertheless of interest to string theory, because, as mentioned above, pp-waves are exact classical solutions in string theory. Furthermore, they too can have interesting duals.

The most obvious way to prove the absence of black holes is to examine the global causal structure of a general pp-wave. Although it turns out to be a rather formidable task to examine the causal structure in full generality, in the ensuing paper [14] we will discuss specific examples and some of the causal properties we expect the general pp-wave to carry. It is somewhat simpler to concentrate on just the plane waves. Indeed, the causal structure of certain plane waves has recently been studied by Marolf and Ross [15], who use the approach introduced by Geroch, Kronheimer, and Penrose [16], which is based on completing the spacetime by "ideal points" reflecting its causal structure. Marolf and Ross demonstrate that for homogeneous plane waves, the conformal boundary consists of a one-dimensional null line plus two points corresponding to future and past infinity. This result agrees with and generalizes that of [17], who obtain the asymptotic structure of the BMN plane wave by conformally mapping it into the Einstein static universe. In the examples studied thus far it is clear that the entire spacetime manifold is in the causal past of infinity, thereby precluding the presence of event horizons.

While the study of pp-wave causal structure has not been completed in full generality, in this paper, we shall content ourselves with examining the more limited (but, from string theory point of view, perhaps the most interesting) aspect of causal structure, which can be addressed in generality. Specifically, we will ask the question *can pp-waves admit horizons?* As revealed above, we will argue that the answer is *no*. This, however, does not mean that there cannot exist black hole/string solutions which are asymptotically plane or pp-wave, though they do not respect the plane or pp-wave symmetries everywhere. We offer a particular simple example in section 5, but work is underway to find more physically interesting solutions.

The outline of this paper is as follows. In the following short section, we review certain basic aspects of plane wave and pp-wave spacetimes, mainly with the view of setting up notation, and offer a definition of event horizons and asymptopia in spacetimes which are not asymptotically flat. We will then motivate an argument for the absence of a horizon in the plane wave spacetime, by showing that any point in the spacetime can communicate to arbitrarily large distances using a symmetry argument. Section 4 presents the no-horizon argument for pp-waves. This is distinct from the arguments presented in section 3, but it simultaneously provides an alternate proof that plane waves can't admit event horizons. While, from the point of view of constructing interesting black hole solutions in pp-waves, up to here our results were negative, in section 5 we try to remedy this by discussing generalizations of pp-waves which would admit event horizons. Finally, we end in section 6 with a more general discussion.

2. Terminology and definitions

To pave the way for arguing why pp-waves cannot admit event horizons, we first explain what are pp-waves and plane waves by writing the corresponding metrics. We then discuss what it would mean for these metrics to admit black holes, and offer a criterion for absence of black holes. In the subsequent sections, we use this criterion to argue that pp-waves do not admit horizons. Note that we shall be concerned with physical spacetimes i.e., ones which are solutions to the Einstein-Hilbert action with matter content satisfying appropriate energy conditions.

2.1 Metrics

To set the notation and re-emphasize terminology, we will write explicitly three classes of spacetimes, in decreasing generality. The *pp-wave* spacetimes are defined as spacetimes admitting a covariantly constant null Killing field. The most useful ones[2] can be writ-

[2]Generically, a background admitting a covariantly constant null Killing field can have non-vanishing $g_{ux^i}(u, x^i)$ components of the metric. Also there is no requirement that the transverse space be flat; for vacuum solutions we could have easily considered Ricci flat transverse metrics. While our arguments are expected to hold for these cases, we will restrict our discussion to metrics of the form presented in eq. (2.1).

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ten as

$$ds^2 = -2\,du\,dv - F(u, x^i)\,du^2 + dx^i\,dx^i \tag{2.1}$$

where the vacuum Einstein's equations dictate that F satisfy the transverse Laplace equation for each u. F, however, can be an arbitrary function of u.

Plane wave spacetimes are those where this harmonic function is in fact quadratic, $F(u, x^i) = f_{ij}(u)\,x^i x^j$ in (2.1), so that plane waves can be written as

$$ds^2 = -2\,du\,dv - f_{ij}(u)\,x^i x^j\,du^2 + dx^i\,dx^i \tag{2.2}$$

Here, $f_{ij}(u)$ can be any function of u, subject to the constraint that for each u, f_{ij} is symmetric and, for vacuum solutions, traceless. As suggested by the name, these metrics have an extra "plane" symmetry, which contains the translations along the wave-fronts in the transverse directions. This can be seen explicitly by casting (2.2) into the Rosen form,[3]

$$ds^2 = -2\,dU\,dV + C_{ij}(U)\,dX^i\,dX^j \tag{2.3}$$

The *homogeneous plane waves* further specialize (2.2) by taking out f's dependence on u,

$$ds^2 = -2\,du\,dv - f_{ij}\,x^i x^j\,du^2 + dx^i\,dx^i \tag{2.4}$$

The BMN plane wave metric [5], found earlier by [19], belongs to this last class, for the special case $f_{ij} = \mu^2 \delta_{ij}$, and $u \equiv x^+$, $v \equiv x^-$ in their notation.

In all the aforementioned spacetimes we have a covariantly constant null Killing vector given as $\xi^a = (\partial/\partial v)^a$. The fact that this is a null Killing vector is obvious from the metric, while its being covariantly constant may be inferred from the vanishing of the Christoffel symbols ${\Gamma^v}_{\mu v}$.

2.2 Event horizons and asymptopia

Black holes are defined as regions of spacetime inside event horizons. Hence, to show that a particular class of spacetimes cannot represent black holes, we need to show that these spacetimes do not admit event horizons. However, in order to do so, we first need to specify a suitable definition of an event horizon.

In asymptotically flat spacetimes, an event horizon is defined simply as the boundary of the causal past of the future null infinity, $\mathcal{H} = \partial I^-[\mathcal{I}^+]$, where the past I^- of future null infinity $\mathcal{I}^+$ is defined as the union of the pasts of all the points P at infinity $\mathcal{I}^+$, i.e., $I^-[\mathcal{I}^+] \equiv \cup I^-(P) : P \in \mathcal{I}^+$. Physically, this just says that an asymptotic observer can't see inside black holes. However, as is well-known, when the spacetime in question is not asymptotically flat, this simple definition may not work. First, the notion of asymptopia may be more murky. For instance, as in the case of closed FRW universe, there can be a big crunch, so that there does not exist any asymptotic region at all. Similarly, for some of

[3]Typically, this metric is not geodesically complete because of coordinate singularities, but the Brinkman form (2.2) does cover the full spacetime. The coordinate transformation from one form into the other is given e.g., in [18]. For metric of the Brinkman form $ds^2 = -2dudv - f(u)x^2du^2 + dx^2$, the coordinate transformation $\{u = U, x = h(U)X, v = V + \frac{1}{2}h(U)h'(U)X^2\}$ where $h(U)$ satisfies $h''(U) + f(U)h(U) = 0$, casts this metric into the Rosen form $ds^2 = -2dUdV + h(U)^2dX^2$.

the presently-studied pp-waves, the "asymptotic" region may be singular if $F(u,x^i) \sim x^p$ for some $p > 2$ in the pp-wave metric (2.1); or the spacetime may terminate at finite u if $F(u,x^i) \to \infty$ as $u \to u_\infty < \infty$.

We will therefore adopt a more universal definition of a black hole, or rather the absence of black hole, which, instead of requiring that any point in the spacetime is "visible" (i.e., causally connected) to asymptotic observer, merely requires that any point in the spacetime is visible to an observer who is "arbitrarily far." This last phrase needs a bit more qualification. One might naively try to define "arbitrarily far" by "some spatial coordinate getting arbitrarily large," but this is too glib since it is a coordinate-dependent statement. First of all, if the coordinates don't cover the entire spacetime, reaching arbitrarily large values of the coordinates would merely indicate coming closer to the boundary of our coordinate patch, not the spacetime. Also, for all geodesics which don't terminate at a singularity, the affine parameter gets arbitrarily large; but in the case of pp-waves, this will be one of the coordinates, u, which has a spatial component.

The first objection can be bypassed in the case of pp-waves: the coordinate patch of the metric (2.1), with all the coordinates ranging form $-\infty$ to ∞, does cover the full spacetime. The second objection might be mollified by noting that u plays the role of time rather than a spatial coordinate, but that does not suffice. Specifically, there always exists a geodesic, for which $\dot{u} = 0$, and v is the affine parameter, so that from any point (u_0, v_0, x_0^i) in the spacetime, we can causally communicate to $(u_0, v \to \infty, x_0^i)$.[4]

Since this point is part of $\mathcal{I}$, the null infinity, one might be tempted to argue straight-off that there can't be event horizons in spacetimes with a null Killing field. The reason we do not wish to do so is that we don't want to preclude horizons stretched along the v direction, (thus respecting the null symmetry), but separating say, a region from which no causal curve can reach arbitrarily large transverse distance, x^i. (Also, there are counter-examples, such as the black holes studied by [20, 21].)

Given the above considerations, we will adopt the following criterion for absence of black holes in pp-wave spacetimes.

Definition. *A pp-wave spacetime does not admit an event horizon iff from any point in the spacetime, say (u_0, v_0, x_0^i), there exists a causal curve to some point $(u_1, v_\infty, x_\infty^i)$, where $u_1 > u_0$ is arbitrary, while $v_\infty, x_\infty^i \to \infty$.*

The important aspect is that not just v, but also at least one of the transverse coordinates, x^i, gets arbitrarily big along a causal curve. We will in fact use a stronger version of this criterion; namely, we will require $u_1 = u_0 + \varepsilon$, for arbitrarily small $\varepsilon > 0$. This will allow us to use the criterion in greater generality, in particular even in the cases where our spacetime terminates at some finite u, i.e., $F(u,x^i) \to \infty$ as $u \to u_\infty < \infty$.

[4]This may be restated in a more covariant fashion as follows: For all spacetimes admitting a null Killing field, the integral curves along these Killing vectors actually describe null geodesics. To see this, denote the Killing vector by ξ^a. Then $\xi^a \nabla_a \xi^b = \xi^a \nabla^b \xi_a = \frac{1}{2}\nabla^b(\xi^a \xi_a) = 0$, where the first equality follows from the definition of Killing vector, the second from product rule, and the last from ξ^a being null and thus having a constant norm. But $\xi^a \nabla_a \xi^b = 0$ is just the geodesic equation, with ξ^a being the tangent vector to the geodesic.

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One more side comment on terminology is in order: We are using the term *event horizon* in an unconventional (generalized) way, as defined above, rather than as something fundamentally related to $\mathcal{I}$. However, it seems likely that if any part of spacetime is visible to an observer who is arbitrarily far, it will also be visible to an asymptotic observer. Also, in the present discussion, *horizon* is used as shorthand for "event horizon" as defined above. It is perhaps worth stressing that there of course are Rindler horizons in pp-waves, just as there are Rindler horizons in e.g. the flat spacetime. These, however, are rather trivial, and don't carry any globally special properties. In particular, they do not bound a black hole.

3. Heuristic motivation for no horizons in plane wave

In the present section we will first give a heuristic motivation for no horizons in plane wave, and then try to argue that in plane wave spacetimes, causal communication from a given point in the spacetime to asymptotically large distances is always possible. As discussed above, this automatically precludes the presence of horizons.

A heuristic argument for the absence of black holes as plane waves is as follows. As shown originally by Penrose [1] in the context of classical general relativity, and later extended to supergravity by [22], a plane wave spacetime can be viewed as a Penrose limit of some spacetime. In this limit, one zooms arbitrarily close to a null geodesic and reexpands the transverse directions — a procedure analogous to obtaining a tangent space by zooming in to a point in a manifold — so that the only nontrivial information which survives is the 1-dimensional structure along the geodesic, parameterized by its affine parameter. The "blowing up" of the transverse directions gives rize to the covariantly constant null Killing field mentioned above. This zooming and reexpanding effectively washes out most of the global information contained in the spacetime in all directions excepting that along the null geodesic. In particular, the limit retains local information about the spacetime, albeit in a more general fashion than the tangent space, but at the expense of losing global information such as that pertaining to event horizons.

In the following subsection, we will illustrate this point with a specific example, the four dimensional Schwarzschild black hole. We will then give a more rigorous argument, essentially based on the symmetries of the plane wave spacetimes.

3.1 Penrose limits of black hole spacetimes

We demonstrate that Penrose limits of black hole spacetimes do not retain information about the event horizon (cf. [23], for considerations of Penrose limits in AdS-Schwarzschild spacetimes). While this is somewhat obvious from the preceding discussion, it may nevertheless serve as an intuition-building exercise. Consider, for instance, the asymptotically-flat Schwarzschild black hole. The causal structure is as given by the Penrose diagram of figure 1. Since the Penrose limit requires us to consider the neighbourhood of null geodesics, let us see what sorts of null geodesics are allowed in the spacetime.

First of all, it is clear that there are radially infalling geodesics, such as $g1$ of figure 1, which describe the trajectory of a photon falling into the black hole. There can also be ones

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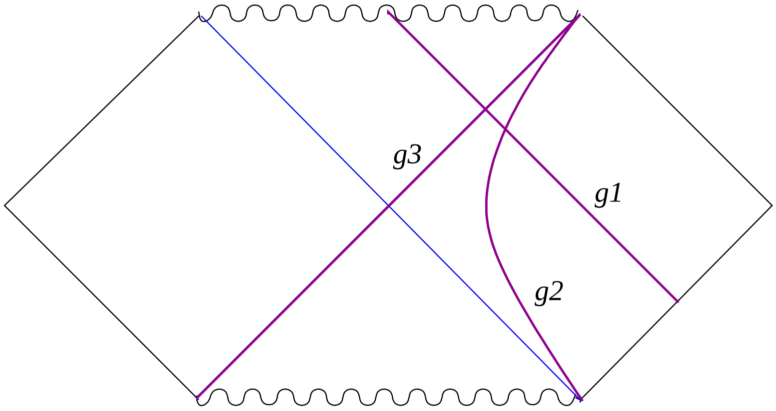

Figure 1: Penrose diagram of the Schwarzschild black hole. The curves $g1$, $g2$, and $g3$ describe different null geodesics which we might consider for taking the corresponding Penrose limit.

with angular momentum (which in the $r - t$ plane of the Penrose diagram would appear timelike), which likewise fall into the singularity. These geodesics intersect the horizon at a single point and terminate at finite value of affine parameter upon hitting the singularity. The resulting plane wave spacetime will have a singularity reflecting this, and in fact the spacelike singularity of the Schwarzschild black hole will be converted to a "cosmological" null singularity. From this construction it is clear that the resulting spacetime will not have a horizon since we keep only a small region close to the point on the horizon where the geodesic intersects the same. In particular, this geodesic would be completely insensitive to, for instance, a null shell which might fall into the black hole later, thus shifting the position of the event horizon of the original spacetime.

A second class of null geodesics are those which are carrying some angular momentum, but staying put at constant values of the radial coordinate, such as the curve labeled by $g2$ in figure 1. This physically corresponds to photon orbits in the black hole spacetime. For the four dimensional Schwarzschild black hole this happens to be at $r = 3M$. However, the neighbourhood of this region is completely smooth and the resulting Penrose limit is just the flat space.

The last interesting geodesic which may be considered is one which is sitting at the horizon i.e., $r = 2M$, and some constant angle $\Omega = \Omega_0$; this is labeled by $g3$ in figure 1. This would be the geodesic just skimming the horizon and one would be most tempted to consider this as the one which can lead to an interesting spacetime in the Penrose limit. This geodesic also leads to a flat space in the Penrose limit for reasons similar to the previous case.

To summarize, Penrose limits of black hole spacetimes are incapable of retaining the global structure of the event horizon. We note in passing that we are here strictly considering Penrose limits of given spacetimes which has a well-defined algorithmic prescription. It is less clear whether there exist other limiting procedures (probably double/mutiple scaling limits), wherein we start with a spacetime with a horizon and end up with a resulting simple spacetime (some analog of plane wave), whilst retaining interesting information about the global causal structure.

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3.2 Geodesics and symmetries in general plane waves

Let us consider the general plane wave metric (2.2). As can be easily checked, the null geodesics are given by

$$\ddot{x}^i + \sum_j f_{ij}(u)\, x^j = 0 \tag{3.1}$$

and the integral constraint

$$v = \frac{1}{2} \sum_i x^i\, \dot{x^i} + v_0\,, \tag{3.2}$$

where v_0 is an arbitrary integration constant which is fixed by the initial conditions. The null Killing field $(\partial/\partial v)^a$ implies that $\dot{u}$ is a constant of motion, so that we can take u to be the affine parameter along the geodesic, and the derivative $\dot{} \equiv d/du$.

Let us now make the following simple observation: Under the constant rescaling $x^i \to \lambda\, x^i$, $v \to \lambda^2\, v$, the metric $ds^2 \to \lambda^2\, ds^2$ remains physically the same (only the "units" get rescaled). Therefore any geodesic

$$\{u, v(u), x^i(u)\} \longrightarrow \{u, \bar{v}(u), \bar{x}^i(u)\} \equiv \{u, \lambda^2\, v(u), \lambda\, x^i(u)\} \tag{3.3}$$

remains the same under this rescaling. Note that this is exactly what we would expect from the geodesic equation: since (3.1) is linear in $x^i(u)$, we are free to rescale x^i; and from (3.2), v is rescaled as $(x^i)^2$.

This rescaling freedom suggests that if a geodesic can make it to some distance x^i, it can make it to arbitrarily large x^i, so there couldn't be a horizon at any finite value of x^i. Note that this is already obvious for v, since $(\partial/\partial v)^a$ is a Killing field, so no value of v can be physically distinguished from any other.

While these arguments were mostly motivational, in the next section we shall present more rigorous proof of nonexistence of event horizon in pp-waves, which include plane waves as a subset.

4. No horizons in pp-waves

In the above, we have shown that there can be no horizons in a generic plane wave. Let us now ask a more general question, namely, *can there be horizons in a pp-wave?*

We shall be working with the pp-wave metric (2.1), but since we will be interested in curves which reach large transverse directions x_i, it is more convenient to rewrite (2.1) in spherical coordinates $\{r, \Omega\}$, where only r can get large. The d-dimensional pp-wave metric then becomes

$$ds^2 = -2\, du\, dv - F(u, r, \Omega)\, du^2 + dr^2 + r^2\, d\Omega_{d-3}^2\,. \tag{4.1}$$

As indicated above, in order to demonstrate the absence of horizons, it suffices to show that from *any* point $p_0 = (u_0, v_0, r_0, \Omega_0)$ of the spacetime, there exists a causal curve $\mathcal{C}$ which reaches arbitrarily large values of r and v in arbitrarily small Δu. Below, we will first try to construct such curves explicitly, and then offer a more elegant proof.

As apparent from (4.1), any causal curve $\mathcal{C}(\lambda)$ must satisfy

$$-2\dot{u}\dot{v} - F(u, r, \Omega)\dot{u}^2 + \dot{r}^2 + r^2\, \dot{\Omega}^2 \leq 0 \tag{4.2}$$

where $\dot{} \equiv d/d\lambda$. Then $\mathcal{C}_0(\lambda) \equiv \{u(\lambda) = u_0, v(\lambda) = \lambda + v_0, r(\lambda) = r_0, \Omega(\lambda) = \Omega_0\}$ is a causal (and in fact null) curve which reaches arbitrarily large values of v. However, it stays at constant r. Since all curves with $\dot{u} = 0$ are simply related to $\mathcal{C}_0$, let us now consider curves with $\dot{u} > 0$. This will be necessary in order for the curve to reach arbitrarily large r. We can then rewrite the causal relation (4.2) simply as

$$-2\,\dot{v} - F(u, r, \Omega) + \dot{r}^2 + r^2\,\dot{\Omega}^2 \leq 0 \tag{4.3}$$

where now $\dot{} \equiv d/du$. Let $\mathcal{C}(u) = \{v(u), r(u), \Omega(u)\}$ be a curve such that

$$\begin{aligned}
\dot{\Omega} &= 0 \quad \Rightarrow \quad \Omega = \Omega_0 \\
\dot{r}^2 &= \alpha F(u, r, \Omega) \\
2\dot{v} &= (\alpha - 1) F(u, r, \Omega) = \left(1 - \frac{1}{\alpha}\right) \dot{r}^2
\end{aligned} \tag{4.4}$$

where α is some constant, to be chosen later, and the initial conditions are given by p_0. Now, if $\Delta u \ll 1$ along the full curve, we can approximate the spacetime region through which such a curve propagates by replacing $F(u, r, \Omega)$ with $F(u_0, r, \Omega)$. Furthermore, since our curve stays at fixed $\Omega = \Omega_0$, we can "freeze" that dependence in F as well. Thus, letting $f(r) \equiv F(u_0, r, \Omega_0)$, our curve $\mathcal{C}$ is arbitrarily well approximated by

$$\begin{aligned}
\dot{\Omega} &= 0 \quad \Rightarrow \quad \Omega = \Omega_0 \\
\dot{r}^2 &= \alpha\, f(r) \\
2\dot{v} &= (\alpha - 1)\, f(r) = \left(1 - \frac{1}{\alpha}\right) \dot{r}^2\,.
\end{aligned} \tag{4.5}$$

Since $\mathcal{C}$ is constructed so as to satisfy (4.3), it is clearly a causal curve. Hence we only need to show that $\mathcal{C}$ exists and can reach arbitrarily large r and v for some choice of α. But (4.5) is a first order system, which we can just integrate forward; the requirement for the existence of the solution is that both $\dot{r}^2$ and $\dot{v}$ remain non-negative. If $|f(r)|$ is bounded from below, we can pick α such that the radial velocity of the curve is always positive, and $\mathcal{C}$ reaches arbitrarily large r.

Construction of a suitable curve is going to be more problematic in regions where $f(r)$ changes sign. To specify the curve completely we first pick the sign of α depending on the sign of $f(r)$ near $r = r_0$. Now suppose $f(r)$ passes through zero for some $r = r_1 > r_0$. At r_1 we also flip the sign of α and continue with the construction of the causal curve. In other words, in each interval in r where $f(r)$ doesn't change sign, we can solve the equation $\dot{r}^2 = \alpha|f(r)|$, and then patch the outgoing pieces together. Of course, such a curve will be continuous but will have discontinuities in its second derivatives.

While this explicitly constructs a causal curve, the technique used to construct the same is not quite elegant. In particular, it requires us to separate the spacetime into various regions depending on the sign of $f(r)$, construct a causal curve which reaches maximal r in each specific region, and then patch the pieces together. To avoid this cumbersomeness, we will now present an alternate proof which is more universal.

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- Pick any point $p_0 = (u_0, v_0, r_0, \Omega_0)$ in the spacetime, and any r_∞, v_∞ (which will represent the arbitrarily large distances that we want to reach by a causal curve). To prove the absence of horizons, we want to show that there exists a point $p_\infty = (u_0 + \varepsilon, v, r_\infty, \Omega)$, with some Ω, $v \geq v_\infty$, and ε arbitrarily small, which lies on a causal curve γ from p_0.

- Pick a constant, F_0, such that $F(u, r, \Omega) \geq F_0 \;\forall r \in (r_0, r_\infty)$. This will clearly be possible if $F(u, r, \Omega)$ is not singular in this region, but we can generalize the proof for singularities as well. We will discuss the existence of F_0 below; but for the moment, we will assume that F_0 does exist.

- Now, consider the fiducial metric,

$$ds_0^2 = -2\, du\, dv - F_0\, du^2 + dr^2 + r^2\, d\Omega_{d-3}^2\,. \tag{4.6}$$

  We want to claim that any curve γ which is causal in ds_0^2 is also causal in ds^2. But this is easily shown:[5] Any curve γ which is causal in ds_0^2 must satisfy $-2\dot{u}\dot{v} - F_0\dot{u}^2 + \dot{r}^2 + r^2\dot{\Omega}^2 \leq 0$, but since $F(u, r, \Omega) \geq F_0$, this automatically implies that (4.2) is also satisfied. This means that if we can find a curve γ from p_0 to p_∞ which is causal in ds_0^2, we are done.

- But that is also easy. To find $\gamma(\lambda)$ such that

$$\begin{aligned}
&\gamma(0) = (u_0, v_0, r_0, \Omega_0)\\
&\gamma(1) = (u_0 + \varepsilon, v \geq v_\infty, r_\infty, \Omega_0)\\
&\text{and} \qquad 0 \geq -2\,\dot{u}\,\dot{v} - F_0\,\dot{u}^2 + \dot{r}^2 + r^2\,\dot{\Omega}^2
\end{aligned} \tag{4.7}$$

  let $\gamma(\lambda)$ be given by e.g.

$$\begin{aligned}
u(\lambda) &= u_0 + \lambda\,\varepsilon\\
v(\lambda) &= v_0 + \lambda\,\Delta v\\
r(\lambda) &= r_0 + \lambda^2\,\Delta r\\
\Omega(\lambda) &= \Omega_0
\end{aligned} \tag{4.8}$$

  with ε and $\Delta r \equiv r_\infty - r_0$ fixed, and Δv to be chosen so as to satisfy the causality condition of (4.7), namely $2\varepsilon(\Delta v + F_0\varepsilon) \geq 4\Delta r^2\lambda^2$ for $0 \leq \lambda \leq 1$. Let

$$\Delta v = \mathrm{Max}\left\{v_\infty - v_0, \frac{2\Delta r^2}{\varepsilon} + |F_0|\varepsilon\right\} \tag{4.9}$$

  Then $\gamma(\lambda)$ satisfies all the requirements of (4.7); in particular, it is causal in ds_0^2. As argued above, this also means that it is causal in the pp-wave spacetime ds^2.

[5]We will now revert back to λ parameterizing our curve, so $\dot{} \equiv d/d\lambda$.

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- In fact, this is easy to see, since ds_0^2 is just the flat spacetime. Explicitly, if $F_0 > 0$, consider the coordinate transformation $u = \frac{1}{\sqrt{F_0}}(t-z)$ and $v = \sqrt{F_0}\, z$; while if $F_0 < 0$, consider the coordinate transformation $u = \frac{1}{\sqrt{-F_0}}\,(t-z)$ and $v = \sqrt{-F_0}\, t$. In both cases, the metric (4.6) becomes $ds_0^2 = -dt^2 + dz^2 + dr^2 + r^2\, d\Omega_{d-3}^2$, which is clearly the d-dimensional Minkowski spacetime. But we know that Minkowski spacetime has no horizons, so that from any point, there exists a causal curve which can attain arbitrarily large values of z and r.

Thus, we have found a causal curve starting from an arbitrary point p_0 of the pp-wave spacetime and attaining arbitrarily large values of the coordinates. Hence no point p_0 can be inside an event horizon, so that there can't be black holes. The only step which still needs to be discussed is the existence of F_0, to which we turn next.

Let us first concentrate on the class of pp-waves which are solutions to vacuum Einstein's equations. Since the Einstein tensor is given by $G_{uu} = \frac{1}{2}\nabla_T^2 F$, where ∇_T^2 is the transverse laplacian, $F(u,r,\Omega)$ of (4.1) must satisfy the transverse Laplace equation, $\nabla_T^2 F = 0$. This is a very remarkable result, since this implies that, due to the linearity of Laplace equation, we may superpose the solutions. In particular, we can decompose F in terms of the generalized $(d-3)$-dimensional spherical harmonics $Y_L(\Omega)$, where $L \equiv \{\ell, m, \ldots\}$:

$$F(u,r,\Omega) = \sum_L \left\{ f_L^+(u)\, r^\ell\, Y_L(\Omega) + f_L^-(u) r^{-(d-4+\ell)}\, Y_L(\Omega) \right\}. \tag{4.10}$$

Therefore along our curve γ, this becomes

$$F(u_0,r,\Omega_0) = \sum_\ell \left\{ f_\ell^+\, r^\ell + f_\ell^-\, r^{-(d-4+\ell)} \right\}. \tag{4.11}$$

We see that there can be singularities at $r=0$ and $r=\infty$. Now, all "singularities" are by definition excluded from our spacetime, in the sense that all points which are part of the physical spacetime must be nonsingular. In particular all starting points p_0 must be of that kind. Thus, if there is a singularity at $r=0$, we must chose $r_0 > 0$. Similarly, we must chose $r_\infty < \infty$. But then in the region of interest, $r_0 \le r \le r_\infty$, F is bounded. This shows that F_0 must always exist for vacuum pp-waves.

What about non-vacuum solutions? This is much more complicated to analyse, since $F(u,r,\Omega)$ can in principle be anything as long as we have the appropriate matter content. *A-priori*, it could obstruct the proposed path of γ by a singularity. For the case of pp-wave solutions that lead to integrable sigma models in light-cone quantization of the world-sheet superstring theory [24, 25], it is possible to see that we can construct causal curves reaching the asymptotic regions of the spacetime.

5. Generalizations admitting horizons

Above, we have seen that for pp-wave spacetimes, namely those admitting a covariantly constant null Killing field, there can't be any event horizons. The existence of this null Killing field played an important role in this observation; in fact, the integral curves of this null Killing vector define null geodesics, so that as we have argued at the beginning, from any point in such a spacetime we can "communicate out to infinity" at $v = \infty$. However, as we also cautioned, this does not automatically guarantee the absence of horizons: without horizons, we should be able to reach infinity in all directions. What is most remarkable is the fact that one is able to communicate causally also in the transverse directions out to large distances.

We therefore want to ask, how many of the properties of pp-waves do we need to relax, in order for the existence of horizons to become possible. We would like to claim that we *can* find black hole solutions admitting a null Killing field, which however is not covariantly constant. While we have no real evidence for this claim, it is easy to see *post facto* that the relaxation of the covariant constancy requirement does lead to spacetimes with an event horizon. In fact, such spacetimes already exist in literature. The simplest such example is the case of traveling waves on a five-dimensional black string as discussed in [21] (cf. [26, 27], and [20] for additional discussions on related issues).

The solution studied in [21] is a solution to the low energy effective action for the heterotic string in five dimensions. The metric and the dilaton for the solution are given by

$$\begin{aligned} ds^2_{\rm str} &= \frac{2}{h(r)}\, du\, dv + \frac{f(r)}{h(r)}\, du^2 + k(r)\, l(r)\left(dr^2 + r^2 d\Omega_2^2\right) \\ e^{4\phi} &= \frac{k(r)\, l(r)}{h^2(r)}\,. \end{aligned} \tag{5.1}$$

The metric has been written in the string frame and the Einstein frame metric is given as $G^{(E)}_{\mu\nu} = g^{(\rm str)}_{\mu\nu} e^{-4\Phi/3}$. The functions appearing in the metric are given by

$$\begin{aligned} f(r) &= 1 + \frac{Q_1}{r} \qquad & h(r) &= 1 + \frac{Q_2}{r} \\ k(r) &= 1 + \frac{P_1}{r} \qquad & l(r) &= 1 + \frac{P_2}{r}\,. \end{aligned} \tag{5.2}$$

There are other fields which need to be turned on for the above metric to solve the equations of motion and we refer the reader to the original source [21] for explicit expressions of the same. By a judicious choice of the charges we can even set the dilaton to be constant; setting $Q_2 = P_1 = P_2$ will suffice for the same.

These spacetimes have an event horizon at $r = 0$, which has a finite area. As $r \to 0$ we see that the radius of the two-sphere takes the constant value $\sqrt{P_1 P_2}$. So we have a finite area and therefore a solution with finite entropy. While the coordinates in which the above metric is written degenerate near the horizon, it is possible to find a set of regular coordinates [21]. The solution is asymptotically flat since the functions appearing in the metric go over to unity for large values of the radial coordinate.

One can imagine recovering a spacetime with a covarinatly constant null Killing vector[6] from the solution given in (5.1), by setting $h(r) = 1$, which requires the choice $Q_2 = 0$. This gives a string frame metric with a regular horizon at $r = 0$ and $(\partial/\partial v)^a$ is a covariantly constant null Killing field, seemingly violating the claims we have made hitherto. However,

[6]We would like to thank Gary Horowitz for bringing this to our attention.

this is illusory. In the physical spacetime i.e., in the Einstein frame metric, we see that $(\partial/\partial v)^a$ is no longer covariantly constant. In addition $r = 0$ is a singular point in the spacetime, for the curvature invariants blow up there. It is interesting, however, that while there is no contradiction in the physical spacetime, there apparently is a violation of our claims in the string frame metric.

One can use the above charged black string solution and generate a solution which is asymptotically plane-wave. The idea is to use the Garfinkle-Vachaspati construction [28, 29] to make the spacetime asymptotically plane-wave. As explained in [21], this is possible for spacetimes which admit a null Killing vector which is hypersurface orthogonal and also show that this procedure leads to spacetimes which have the same set of curvature invariants as the original spacetime.

For the particular case of the asymptotically flat black string, this construction implies that we can add a term $(\Psi(u,r,\Omega)/h(r))du^2$ to the metric appearing in (5.1); the resulting metric is a solution to Einstein's equations with all other fields unaltered, so long as $\Psi(u,r,\Omega)$ is a harmonic function in the transverse space. In particular, we can have

$$\Psi(u,r,\Omega) = \sum_{\ell} \left\{ \psi_\ell^+(u)\, r^\ell\, Y_{\ell m}(\Omega) + \psi_\ell^-(u)\, r^{-(1+\ell)}\, Y_{\ell m}(\Omega) \right\}, \tag{5.3}$$

with arbitrary functions $\psi_\ell^\pm(u)$. All spacetimes with $\Psi(u,r,\Omega) \sim r^\ell$ for $\ell > 2$ suffer from singular behaviour at $r = \infty$, while those with $\Psi(u,r,\Omega) \sim r^{-(\ell+1)}$ with $\ell > 0$ are singular at $r = 0$. By singular we mean that there are divergent tidal forces on finite-sized observers. Addition of $\psi_0^+(u)$ and $\psi_1^+(u) r Y_{1m}(\Omega)$ lead to spacetimes which are diffeomorphic to the original, and the monopole solution leads to a regular spacetime, as was demonstrated by [21]. These cases are the most uninteresting ones as far as constructing a black hole spacetime which is asymptotically plane wave. The interesting case therefore is the case when $\Psi_{\rm plane}(u,r,\Omega) = \psi_2^+(u) r^2 Y_2(\Omega) = \psi_2^+(u)(2z^2 - x^2 - y^2)$, reverting back to cartesian coordinates. Now it is clear that $ds^2_{\rm str} + \frac{\Psi_{\rm plane}(u,r,\Omega)}{h(r)} du^2$ is an asymptotically plane wave spacetime, whilst retaining the regular black hole horizon at $r = 0$. These statements of course remain true in the Einstein frame as well. The string frame metric for the solution is then given as (setting $\psi_2^+(u) = 1$),

$$\begin{aligned} ds^2_{\rm str} &= \frac{2}{h(r)} du\, dv + \frac{f(r) + r^2\left(3\cos^2\theta - 1\right)}{h(r)} du^2 + (k(r) l(r))^2 \left(dr^2 + r^2\, d\Omega_2^2\right) \\ e^{4\phi} &= \frac{k(r)\, l(r)}{h^2(r)}\,, \end{aligned} \tag{5.4}$$

where the functions $f(r)$, $h(r)$, $k(r)$, $l(r)$ are given in (5.2). Once again the Einstein frame metric is $G^{(E)}_{\mu\nu} = g^{(\rm str)}_{\mu\nu} e^{-4\Phi/3}$. The essential trick in constructing the same is that close to the origin, the plane wave is identical to flat space and so given a spacetime which has a horizon at $r = 0$, we can make it asymptotically plane wave whilst keeping the horizon.

The above construction provides an example of a charged black string which is asymptotically of the plane wave form. This naturally begs the question whether there isn't a neutral black string which is asymptotically a plane wave. We cannot "uncharge" the above solution to get a neutral solution, as then the horizon shrinks to zero size; setting any of the charges to be zero causes the horizon to shrink toward the singularity. However, we should be able to take two such solutions with opposite charges and collide them. Since each such black string has a finite horizon area, the area theorem will tell us that the resulting solution ought to have a horizon of finite area. Colliding two such solutions shouldn't change the asymptotics and hence we should have a spacetime which is a neutral black string with a finite entropy and asymptoting to a plane-wave spacetime.

6. Discussion

In the first four sections of this paper, we have established that pp-waves cannot admit event horizons. While this is easily motivated for plane waves, we have provided an alternate argument for the more general pp-waves, which in particular applies to plane waves. For the particular case of plane waves one can argue for the absence of horizons in a more rigorous fashion following the analysis of the causal structure of these spacetimes in [15]. We shall present similar arguments for pp-waves in a future work [14].

The reason for considering pp-waves rather than just plane waves is that for both classes, all the curvature invariants vanish, so that they represent exact classical solutions to string theory. Also, given a plane wave background, one simple deformation of the same is to convert it into a pp-wave background. This follows trivially from the fact that the Einstein's equations for the metric ansatz in (2.1) are linear, enabling superposition of the solutions. In fact, this is the simplest example of the Garfinkle-Vachaspati construction [28, 29]. In a sense, these deformations are similar to exactly marginal deformations in the world-sheet theory, though they are non-normalizable. It is then perhaps somewhat disappointing that these classes of exact classical solutions cannot admit horizons.

However, this does not mean that these solutions cannot be modified to include horizons. After all, on physical grounds, one would expect that if one puts some matter into a plane wave which respects the necessary symmetries, this matter may nevertheless be Jeans-unstable to collapsing. Naively, one may then expect to obtain a black hole. This of course does not conflict with our previous conclusions, since such black holes would break the original symmetries. In particular, the geometry would no longer support a covariantly constant null Killing field. It would be very interesting to study this Jeans instability, and to follow the evolution dynamically, but that is beyond the scope of the present paper.

Instead, in the previous section, we have explored only a mild relaxation of the pp-wave symmetries, namely, keeping the null Killing field, and dropping only the requirement that it be covariantly-constant. Hence, this is not a pp-wave, and the curvature invariants do not vanish. We have presented an explicit solution of a black string with a horizon which is asymptotically a plane wave, in five-dimensions. It would be very interesting to study solutions which do not carry any charges. One strategy as mentioned earlier would be try to collide two oppositely charged asymptotically plane wave black strings. The collision of plane waves is a problem that has been discussed hitherto in [30, 31, 32]. Perhaps it would be possible to extend these discussions to the charged black string discussed above.

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References

[1] R. Penrose, *Any spacetime has a planewave as a limit*, in *Differential geometry and relativity*, pp. 271–275, Reidel, Dordrecht, 1976.

[2] R. Penrose, *A remarkable property of plane waves in general relativity*, *Rev. Mod. Phys.* **37** (1965) 215.

[3] D. Amati and C. Klimčík, *Nonperturbative computation of the Weyl anomaly for a class of nontrivial backgrounds*, *Phys. Lett.* **B 219** (1989) 443.

[4] G.T. Horowitz and A.R. Steif, *Space-time singularities in string theory*, *Phys. Rev. Lett.* **64** (1990) 260.

[5] D. Berenstein, J.M. Maldacena and H. Nastase, *Strings in flat space and PP waves from $N = 4$ super Yang-Mills*, *J. High Energy Phys.* **04** (2002) 013 [hep-th/0202021].

[6] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [hep-th/9711200].

[7] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [hep-th/9802150].

[8] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett.* **B 428** (1998) 105 [hep-th/9802109].

[9] O. Aharony, S.S. Gubser, J.M. Maldacena, H. Ooguri and Y. Oz, *Large-N field theories, string theory and gravity*, *Phys. Rept.* **323** (2000) 183 [hep-th/9905111].

[10] A.A. Tseytlin, *On limits of superstring in $AdS_5 \times S^5$*, hep-th/0201112.

[11] L.A. Pando Zayas and D. Vaman, *Strings in RR plane wave background at finite temperature*, hep-th/0208066.

[12] B.R. Greene, K. Schalm and G. Shiu, *On the hagedorn behaviour of PP-wave strings and $N = 4$ SYM theory at finite R-charge density*, hep-th/0208163.

[13] Y. Sugawara, *Thermal amplitudes in DLCQ superstrings on PP-waves*, hep-th/0209145.

[14] V.E. Hubeny and M. Rangamani, *Causal structures of pp-waves*, hep-th/0211195.

[15] D. Marolf and S.F. Ross, *Plane waves: to infinity and beyond!*, hep-th/0208197.

[16] R. Geroch, E.H. Kronheimer and R. Penrose, *Ideal points in space-time*, *Proc. Roy. Soc. Lond.* **A 327** (1972) 545.

[17] D. Berenstein and H. Nastase, *On lightcone string field theory from super Yang-Mills and holography*, hep-th/0205048.

[18] M. Blau, J. Figueroa-O'Farrill, C. Hull and G. Papadopoulos, *Penrose limits and maximal supersymmetry*, *Class. and Quant. Grav.* **19** (2002) L87 [hep-th/0201081].

[19] M. Blau, J. Figueroa-O'Farrill, C. Hull and G. Papadopoulos, *A new maximally supersymmetric background of IIB superstring theory*, *J. High Energy Phys.* **01** (2002) 047 [hep-th/0110242].

[20] G.T. Horowitz and D. Marolf, *Counting states of black strings with traveling waves*, *Phys. Rev.* **D 55** (1997) 835 [hep-th/9605224].

[21] N. Kaloper, R.C. Myers and H. Roussel, *Wavy strings: black or bright?*, *Phys. Rev.* **D 55** (1997) 7625 [hep-th/9612248].

[22] R. Güven, *Plane wave limits and T-duality*, *Phys. Lett.* **B 482** (2000) 255 [hep-th/0005061].

[23] L.A. Pando Zayas and J. Sonnenschein, *On Penrose limits and gauge theories*, *J. High Energy Phys.* **05** (2002) 010 [hep-th/0202186].

[24] J. Maldacena and L. Maoz, *Strings on PP-waves and massive two dimensional field theories*, hep-th/0207284.

[25] J.G. Russo and A.A. Tseytlin, *A class of exact PP-wave string models with interacting light-cone gauge actions*, *J. High Energy Phys.* **09** (2002) 035 [hep-th/0208114].

[26] M. Cvetič and A.A. Tseytlin, *Solitonic strings and BPS saturated dyonic black holes*, *Phys. Rev.* **D 53** (1996) 5619 [hep-th/9512031]; erratum *ibid.* **D55** (1997) 3907.

[27] M. Cvetič and D. Youm, *General rotating five dimensional black holes of toroidally compactified heterotic string*, *Nucl. Phys.* **B 476** (1996) 118 [hep-th/9603100].

[28] D. Garfinkle and T. Vachaspati, *Cosmic string traveling waves*, *Phys. Rev.* **D 42** (1990) 1960.

[29] D. Garfinkle, *Black string traveling waves*, *Phys. Rev.* **D 46** (1992) 4286 [gr-qc/9209002].

[30] P. Szekeres, *Colliding plane gravitational waves*, *J. Math. Phys.* **13** (1972) 286.

[31] K. Kahn and R. Penrose, *Scattering of two impulsive gravitational plane waves*, *Nature* **229** (1971) 185.

[32] S. Chandrasekhar and B.C. Xanthopoulos, *A new type of singularity created by colliding gravitational waves*, *Proc. Roy. Soc. Lond.* **A 408** (1986) 175.